

# Universal quantum computation in integrable systems

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*Abstract:* Quantized integrable systems can be made to perform universal quantum computation by the application of a global time-varying control. The action-angle variables of the integrable system function as qubits or qudits, which can be coupled selectively by the global control to induce universal quantum logic gates. By contrast, chaotic quantum systems, even if controllable, do not generically allow quantum computation under global control.

In classical mechanics, integrable systems are ones that are dynamically ‘well-behaved.’ Their dynamics are characterized by a set of conserved quantities, and are stable under small perturbations [1]. The quantized versions of integrable linear systems inherit much of the good behavior of their classical counterparts [2-5]. In this paper, we investigate the problem of controlling quantized integrable systems and show that generically, quantized integrable systems can be made to perform universal quantum computation by application of a single, global, time-varying control. Quantum computers are devices that process information using quantum coherence and entanglement. They can build up arbitrarily complicated computations by performing quantum logic operations on a few variables at a time. Techniques of electromagnetic resonance allow one to construct quantum computers from arrays of coupled quantum systems by applying a single, time-varying global control field [6-8]. Here, we show that the same technique can be used to make the quantized versions of integrable systems compute: the action-angle variables of the integrable systems become the logical degrees of freedom of the quantum computer, and a global control

allows the systematic coupling and manipulation of those variables to perform universal quantum computation.

The dynamics of a classical Hamiltonian integrable system can be decomposed into action-angle variables, so that the motion of the system represents a trajectory on a set of tori in a  $2n$ -dimensional phase space. The KAM theorem shows that these trajectories are relatively stable under perturbation [1]. The action variables  $I_j$  have vanishing Poisson brackets with the Hamiltonian  $H$  of the system:  $\{H, I_j\} = 0$ , and so are conserved. Their Poisson brackets with each other also vanish:  $\{I_j, I_k\} = 0$ . The angle variables  $\theta_j$  rotate at frequency  $\omega_j = \partial H / \partial I_j$ . If the  $\omega_j$  are incommensurate, the state of the system covers an invariant torus in phase space.

In the quantized version of an integrable system [2-5], the Hamiltonian and action-angle variables become operators and the Poisson brackets are replaced by commutators. Because they all commute with each other, the Hamiltonian and the action variables can be simultaneously diagonalized and have joint eigenstates  $|i\rangle = |E_i\rangle|i_1\rangle \dots |i_n\rangle$ . Suppose that our system is initialized in such an eigenstate  $|\psi_0\rangle = |E^0\rangle|i_1^0\rangle \dots |i_n^0\rangle$ . Look at the eigenspace  $\mathcal{H}_0$  spanned by eigenstates with eigenvalues that lie within a range  $\Delta E$ ,  $\Delta i_1 \dots \Delta i_n$  of the eigenvalues of  $|\psi_0\rangle$ . Because the KAM theorem tells us that the classical dynamics are periodic and robust under small perturbations, we can approximate the quantum Hamiltonian over  $\mathcal{H}_0$  by

$$H = H_0 + \sum_j \frac{\partial H}{\partial I_j} \Delta I_j + \sum_{jk} \frac{\partial^2 H}{\partial I_j \partial I_k} \Delta I_j \Delta I_k, \quad (1)$$

where  $\frac{\partial H}{\partial I_j} = \omega_j$  as above, and  $\Delta I_j, \Delta I_k$  are harmonic oscillator Hamiltonians restricted to  $\mathcal{H}_0$ . That is, in the subspace spanned by eigenstates in the vicinity of our starting state  $|\psi_0\rangle$ , the quantized version of the integrable system behaves to first order like a collection of uncoupled harmonic oscillators, with couplings and nonlinear behavior that appear at second order.

It is well-known how to make such quantum systems compute [6-8]. The quantized oscillators become the qudits of our quantum computer. By driving at the individual oscillator frequencies  $\omega_j$ , and taking advantage of the second-order nonlinearity, one can induce any desired transformation of the oscillators individually. By driving at the difference frequencies  $\omega_j - \omega_k$ , one induces two-qudit transformations between the  $j$ th and  $k$ th oscillators. Taken together, such one- and two-qudit operations allow one to perform quantum computation.

Make this argument precise. Add a global control field, with a time-varying strength:

$$H(t) = H + \gamma(t)H_c. \quad (2)$$

The state of the system  $|\psi(t)\rangle$  evolves under the time-dependent Schrödinger equation,  $i\partial|\psi\rangle/\partial t = H(t)|\psi\rangle$ . First consider periodic driving,  $\gamma(t) = \gamma_0 \cos \omega t$ , and go to the interaction picture. The state in the interaction frame is  $|\chi\rangle = e^{iHt}|\psi\rangle$ , which obeys the equation

$$i\frac{\partial|\chi\rangle}{\partial t} = \gamma_0 \cos \omega t e^{iHt} H_c e^{-iHt} |\chi\rangle. \quad (3)$$

Write  $H_c = \sum_{ii'} a_{ii'} |i\rangle\langle i'|$  in terms of eigenstates  $|i\rangle$  of  $H$ . In this basis, equation (3) becomes

$$i\frac{\partial|\chi\rangle}{\partial t} = \sum_{ii'} (\gamma(t)a_{ii'}/2) (e^{i(E_i - E_{i'} - \omega)t} + e^{i(E_i - E_{i'} + \omega)t} |i\rangle\langle i'| |\chi\rangle). \quad (4)$$

We go to the rotating-wave approximation by dropping oscillating terms, retaining only those where  $E_i - E_{i'} \pm \omega \approx 0$ . In the rotating-wave approximation,  $|\chi\rangle$  obeys the equation

$$i\frac{\partial|\chi\rangle}{\partial t} = \tilde{H}_c |\chi\rangle, \quad (5)$$

where  $\tilde{H}_c = (1/2) \sum_{ii': E_i - E_{i'} \pm \omega \approx 0} \gamma_0 a_{ii'} |i\rangle\langle i'|$ . The periodic driving then drives on-resonant transitions with Rabi frequency  $\gamma_0 a_{ii'}$  in the co-rotating frame. In  $SU(2)$  notation, if the two energy eigenstates of the on-resonant transition are identified with spin- $z$   $\uparrow, \downarrow$ , then we can drive rotations  $e^{-i\gamma_0 t \sigma_{\hat{j}}/2}$ , where  $\hat{j}$  is a vector in the  $x - y$  plane in the co-rotating frame whose phase is determined by the phase of the sinusoidal driving term. By varying this phase, we can construct any desired  $SU(2)$  transformation in the two-level subspace of the on-resonant transition. Assume that the  $\omega_j$  are incommensurate (for a typical chaotic system, their distribution is Poissonian [2-5]). Varying the frequency of the driving field to drive different transtions affords full and systematic control of the integrable quantum system.

Now analyze the size of the errors introduced by the rotating-wave approximation. To zero'th order in perturbation theory, the energies of the off-diagonal terms oscillate, yielding an average phase of zero. To first order, the off-resonant transitions undergo small, rapid oscillations with frequency  $\Omega = \sqrt{\gamma_0^2 + \Delta\omega^2}$ , where  $\Delta\omega^2 = (E_i - E_{i'} \pm \omega)^2$ , and amplitude  $O(\gamma_0^2/\Omega^2)$ . These are the terms ignored in the rotating-wave approximation: they can be made as small as desired by driving with weaker and weaker fields. The effectiveness of the rotating-wave approximation is justified by the KAM theorem [1]: the effect of the

small perturbations on the action-angle variables is simply to perturb their frequencies. For variables on resonance with the field, the perturbation induces a controllable unitary evolution. For off-resonant variables, the oscillating perturbations average to zero. Because we have assumed the frequencies of the action-angle variables to be incommensurate, the length of the pulses required to perform selective driving to the desired accuracy goes as  $O(n)$ : the time it takes to perform selective driving grows linearly in the size of the system.

The method of selective driving allows one to perform universal quantum control of integrable systems. To perform universal quantum computation, we must be able to control the states of individual action-angle variables and couple pairs of variables efficiently. Begin in the ground state  $|0\rangle \otimes \dots \otimes |0\rangle$ , and drive transitions only in the ground and first excited states  $\ell = 0, \ell = 1$  for each oscillator. These form the qubits of the universal computation. The anharmonic spectrum of the Hamiltonian, together with the assumption that their frequencies are incommensurate, implies that all single qubit transitions have unique frequencies and can be driven selectively using the global drive. So frequency selection allows us to perform arbitrary single qubit  $SU(2)$  transformations on individual oscillators. Driving at the frequency  $\omega_j - \omega_k$ , performs  $SU(2)$  transformations on the  $|10\rangle, |01\rangle$  subspace of the  $j$ 'th and  $k$ 'th oscillator subsystem – i.e., one can continuously ‘swap’ the  $j$ 'th and  $k$ 'th qubits. (Alternatively, because the  $I_j$  all commute with the Hamiltonian, one can emulate NMR quantum information processing [7-8] and use delays between resonant pulses together with the interactions between action variables to effect  $\Delta I_j \otimes \Delta I_k$  operations.) Single-qubit rotations and two-qubit continuous swap operations are universal for quantum computation.

If the control Hamiltonian drives all transitions with equal strength, so that  $a_{ii'}$  is a constant, then we are done: quantum integrable systems can be made to perform universal quantum computation with a single global driving field. In general, however, the transition frequencies  $\gamma_0 a_{ii'}/2$  depend on the states of the other oscillators that are not involved in the single-oscillator and two-oscillator transitions. That is, the transition driven at frequency  $\omega_j$  for the  $j$ 'th actually corresponds to a band of transition frequencies oscillator  $\gamma_0 a_{ii'}$ , where the  $i, i'$  label not only the state of the  $j$ 'th oscillator, but the states of the other oscillators as well. To cope with this variation, make the driving field of the form  $\gamma(t) = \gamma_0(t) \cos \omega t$ , where  $\gamma_0(t)$  is a slowly varying envelope field with bandwidth smaller than the frequency difference  $r\omega_j$  but large enough to cover the band of transition frequencies  $\gamma_0 a_{ii'}$ . Adjust the strength of the component of  $\gamma_0(t)$  at each of these frequencies within

the band to compensate for the variation in the couplings  $a_{ii'}$ . This adjustment can be performed, for example, using optimal control techniques [9-10]. In [11] it is shown that the complexity of the control problem for integrable systems scales polynomially with the number of variables in the system. As in decoupling pulses in NMR quantum computing [7-8], the length of the compensating pulse grows with the number of action-angle variables. In practice, the global control may couple each action-angle variable to a few others, so that only a small variation in transition frequencies need be compensated for. The same technique allows us to compensate for variation in transition frequencies when coupling two oscillators.

Combined with the ability to prepare and measure the state of at least one of the oscillators, the ability to perform continuous time global control allows universal quantum logic via frequency selection. The method is similar to quantum computation via electromagnetic resonance [6-8]. As in NMR quantum computation [7-8], the global nature of the control and coupling leads eventually to the ‘forest of lines’ problem: the number of spectral lines that one wishes to resolve grows with the number of oscillators, so that the selective pulses must become weaker, longer, and more exact in frequency as the number of oscillators increases. More precisely, as the number of action-angle variables  $n$  becomes large, the time it takes to perform each selective pulse goes as  $O(n)$ , so that the time required to perform  $m$  operations scales as  $O(mn)$  rather than as  $n$ . This polynomial slowdown can be resolved if the action-angle variables correspond to variables that are spatially localized, so that the now no-longer global control field can be applied to only a few oscillators at a time.

### *Quantum computation and chaotic systems*

Comparing the control of quantized integrable systems with that of quantized chaotic systems [2-5], we see that the method of resonant control fails. The spectrum of quantized chaotic systems typically obeys a Wigner-Dyson distribution, so that the separation between energies in the spectrum decreases exponentially in the system size. Although we can still use resonance methods to drive transitions between energy eigenstates of the chaotic system and to build up arbitrary unitary transformations, the length of the selective pulses now grows exponentially with the size of the system. For example, we can impose a qubit structure on a chaotic quantum system by labeling the energy eigenstates using binary numbers (e.g., label them sequentially from smallest to largest). However, the spectrum will not obey the simple form of pairwise coupled qubits or qudits as in equation

(1). Instead, the interactions between qubits will be highly non-local: for a typical chaotic system they obey the same statistics as a random Hamiltonian [2-5]. In this case, controlling qubits individually takes an exponentially large time. Although quantized chaotic systems are generically controllable via a time-varying global field [12], their controllability does not mean that quantized chaotic systems can be made to perform universal quantum computation.

If the chaotic system is close to integrable, in the sense that it possesses a natural tensor product structure with weak interactions amongst spectrally resolvable variables, then we can use resonant quantum control selectively to decouple and recouple those variables, leading to universal computation as above. The chaotic nature of the system will introduce noise into the system at a rate given by the Kolmogorov-Sinai entropy: as long as this rate is sufficiently small then bang-bang control or quantum error correction techniques can compensate for it in principle. (Quantum error correction can also compensate for noise induced by the interaction of an integrable system with an environment.) For a fully chaotic, strongly coupled system, however, with no natural tensor product structure, global control does not suffice to perform quantum computation.

*Discussion:* Because integrable systems can be decomposed in terms of action-angle variables, the quantized version of such a system consists to first order of non-interacting quantized harmonic oscillators, in which nonlinearities and couplings enter at second order. The theory of resonant driving then shows that a global time-varying control field can be used selectively to drive individual oscillators and couple pairs of oscillators to perform universal quantum computation. By contrast, quantum chaotic systems do not possess a natural tensor product structure. Although it may be controllable via a global field, there is no obvious way to make a strongly chaotic quantum system perform universal quantum computation.

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